Burgers equation and Fourier Fractal Decimation

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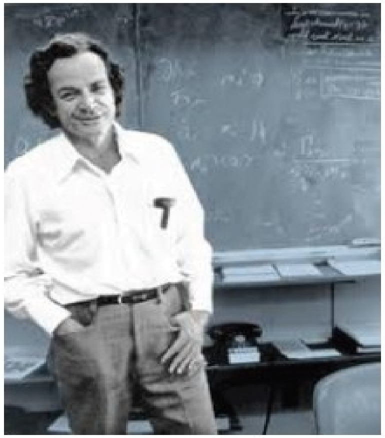
Outline:

1) Why are **Navier-Stokes** equations interesting for Theoretical Physics?

- Strongly non perturbative field Theory (Classical)
- Anomalous Scaling (Non-Gaussian Statistics)



"With turbulence, it's not just a case of physical theory being able to handle only simple cases—we can't do any. We have no good fundamental theory at all." (Feynman, 1979, Omni Magazine, Vol. 1, No.8).

2) Why do we need a model for Navier-Stokes?

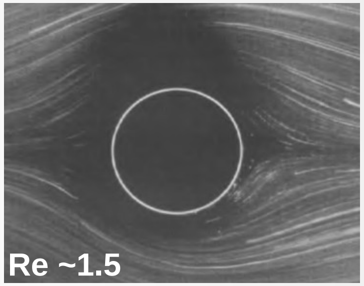
3) Burgers' equation and Fourier Fractal Decimation

Probabilistic description for fully developed Turbulence

Navier-Stokes, (N-S), equations:

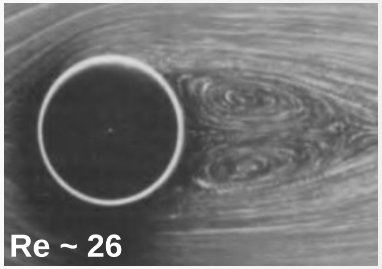
$$\begin{cases} \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t)\cdot\nabla_x \mathbf{v}(\mathbf{x},t) = -\nabla_x p(\mathbf{x},t) + \nu \Delta_x \mathbf{v}(\mathbf{x},\mathbf{t}) + \mathbf{f}(\mathbf{x},t) \\ \nabla_x \cdot \mathbf{v}(\mathbf{x},t) = 0 \end{cases}$$

$$\begin{cases} \hat{t} = t/t_0 \\ \hat{x} = x/l_0 \\ \hat{v} = v/v_0 \end{cases} \qquad \partial_t \hat{v} + \hat{v}\cdot\partial\hat{v} = -\partial\hat{P} + \frac{1}{Re}\partial^2\hat{v} \qquad Re = \frac{l_0 v_0}{\nu} \qquad Re \sim \frac{\hat{v}\partial\hat{v}}{\nu\partial^2\hat{v}}$$

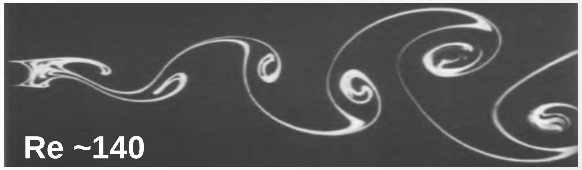


Re

Left-right invariance is broken — ~ 10 — **Recirculating standing eddies**

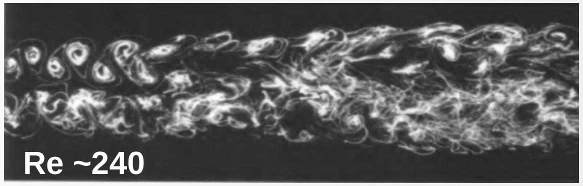


Z-invariance is broken, **discrete time invariance** — **Kàrmàn** street

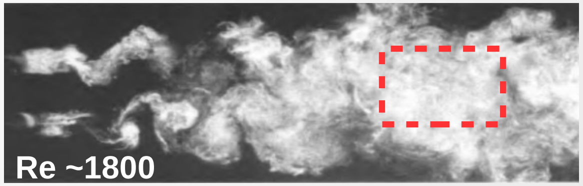


$\sim 10^2$

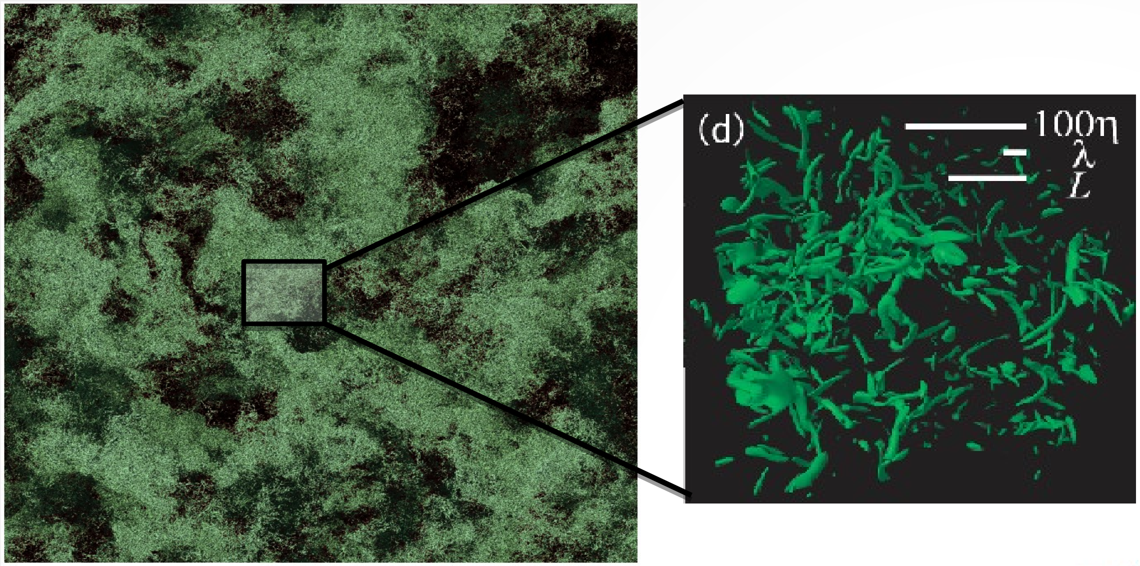
At high Re symmetries are **spontaneously broken** — Flow becomes **chaotic** in its time-dependence



Restored symmetries; (in a statistical sense) — $\sim 10^3$ — Homogeneous-isotropic **fully developed turbulence**



Anomalous Exponents, Small-Scales Intermittency

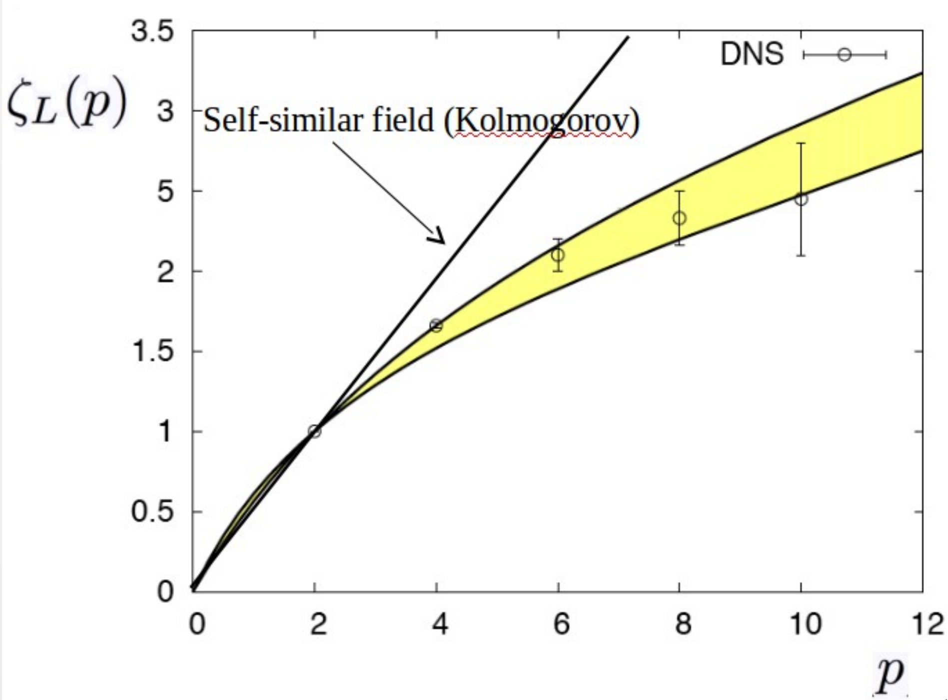


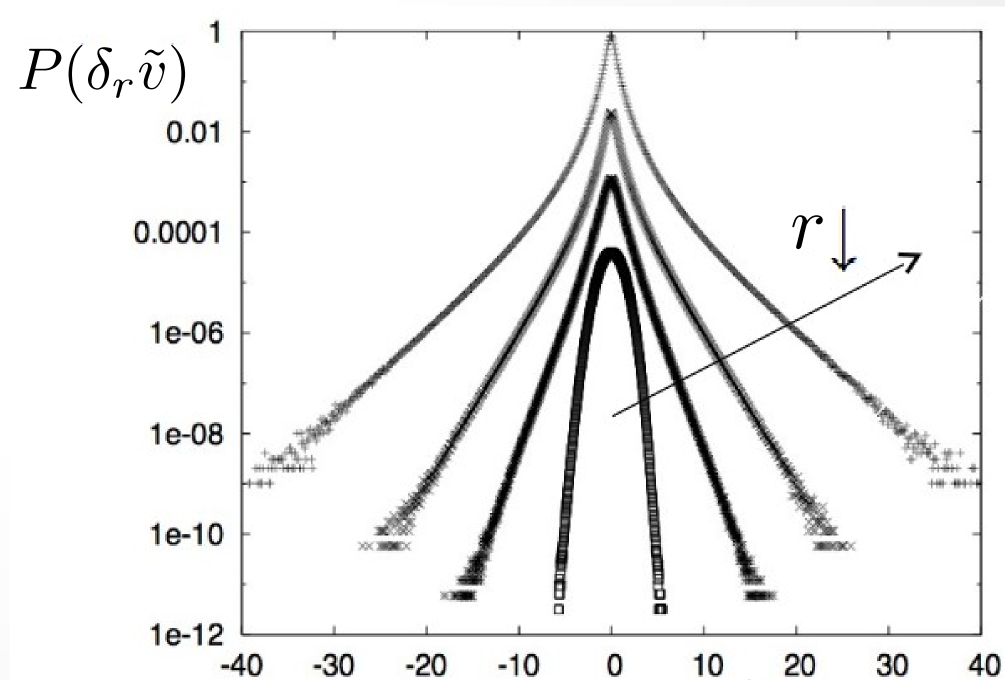
H1) **Restored symmetries** (in a statistical sense).

H2) **Self-similarity** at small scales.

$$S_p(r) =< (\delta_r v)^p > \sim r^{\zeta(p)}$$

$$\delta_r v = v(x+r) - v(x)$$





Benzi, Biferale, Fisher, Lamb and Toschi, JFM (2010).

..a model for Turbulence

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

Burgers' equation

u(t,x): velocity field, depending on a variable of time (***t***), and on a variable of space (***x***) | ***ν***: kinematic viscosity

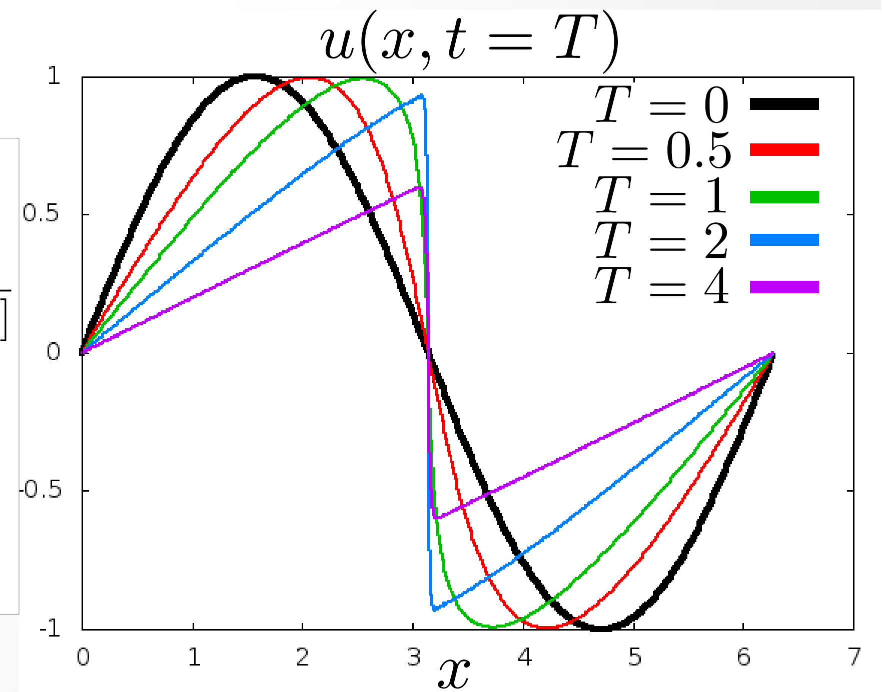
Burgers **produces a singularity**, (**shock**).

Lagrangian observation

$$\begin{cases} u(t,X(t,a)) = u_0(a) \\ X(t,a) = a + tu_0(a) \end{cases} ;\ J(t,a) = \frac{\partial X}{\partial a} = 1 + tu_0'(a)\ ;\ t^* = \frac{1}{-inf_a[u_0'(a)]}$$

Gradient in the Eulerian coordinates

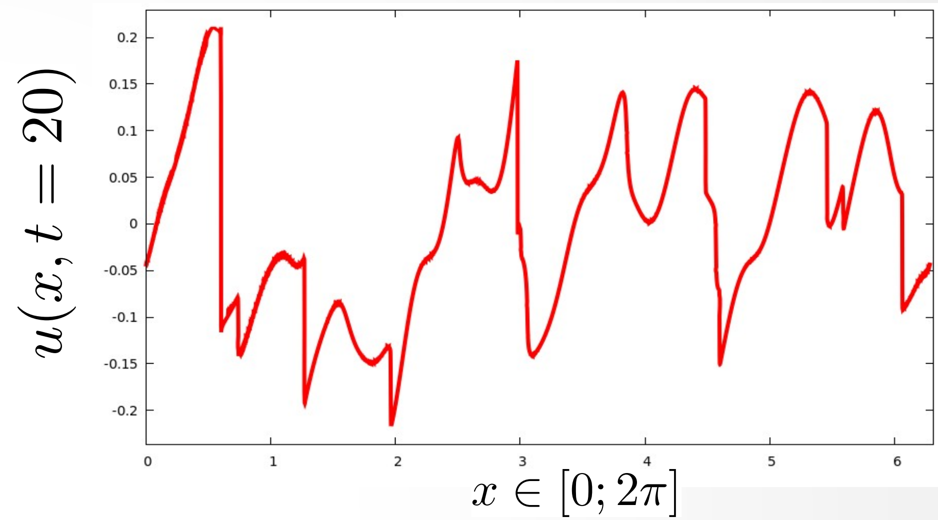
$$\left.\frac{\partial u}{\partial x}\right|_{x^*=a^*} = \left.\frac{\partial u}{\partial a}\right|_{a^*}\left.\frac{\partial a}{\partial x}\right|_{x^*} = u_0'(a)\frac{1}{1+tu_0'(a)} \rightarrow \lim_{t\rightarrow t^*}\frac{u_0'(a)}{1+tu_0'(a)} = \infty$$

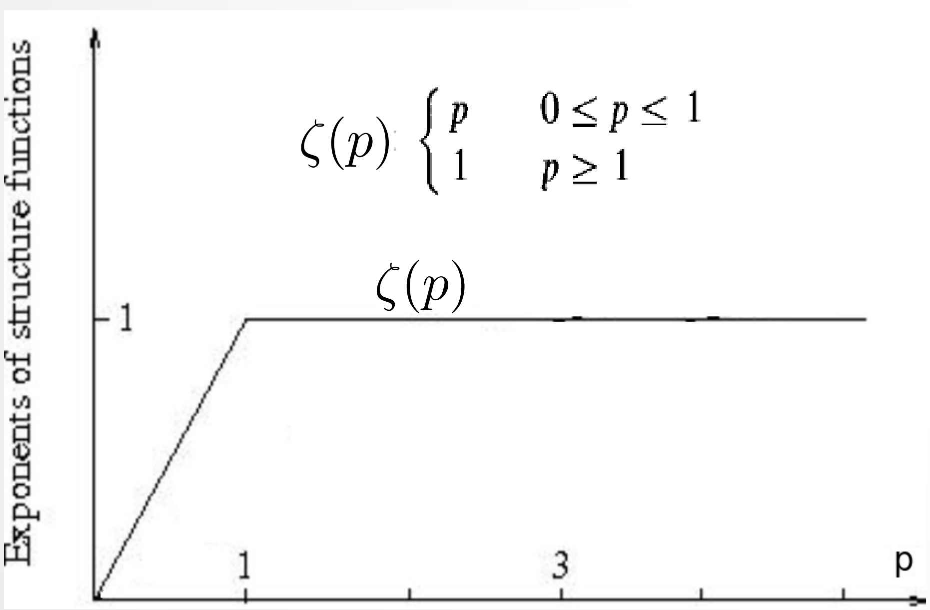


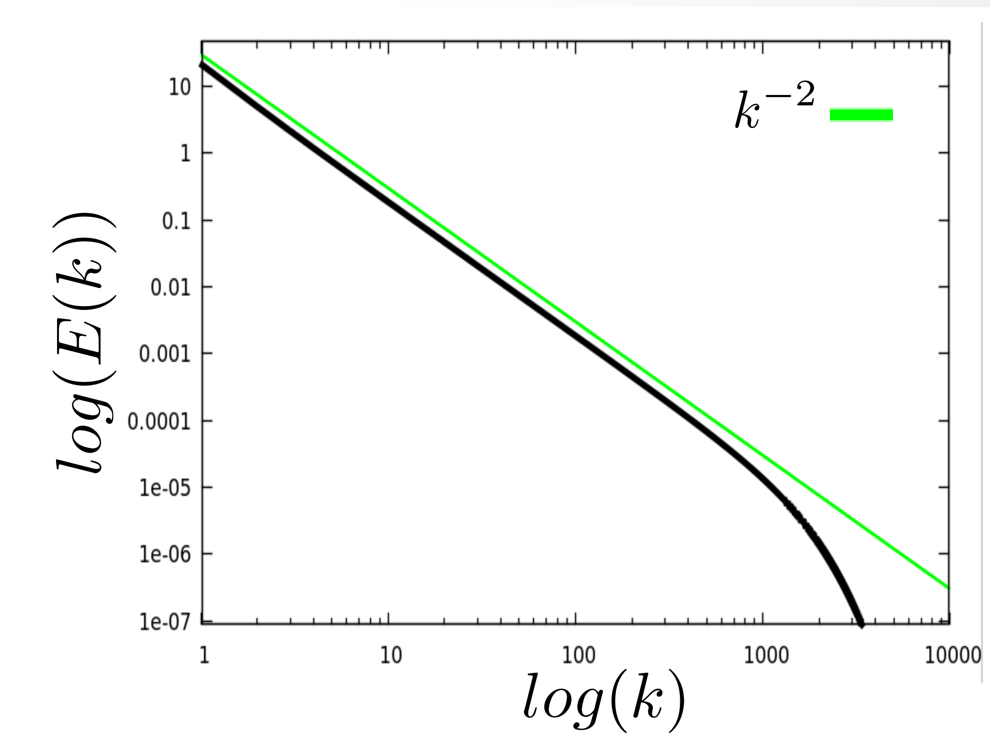
Intermittency on Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

$$S_p(r) = < (\delta_r v)^p > \sim r^{\zeta(p)}$$







how many degrees of freedom are related to the singularity?

..Reduce to learn!

Frisch, Pomyalov, Procaccia, and Ray, Turbulence in non-integer dimensions by fractal Fourier decimation. Phys. Rev. Lett. 108, (2012)

FRACTAL FOURIER DECIMATION

$$u(x,t) = \sum_{k \in Z} e^{ikx} u(k,t) \qquad P_D \cdot u(x,t) = \sum_{k \in Z} e^{ikx} \theta_k u(k,t)$$

$$\theta_{\mathbf{k}} = \begin{cases} 1 \text{ with probability } h_k \\ 0 \text{ with probability } 1 - h_k \,, \end{cases} \quad k \equiv |\mathbf{k}|$$

$$h_k = (k/k_0)^{D-1}, \quad 0 < D \leq 1$$

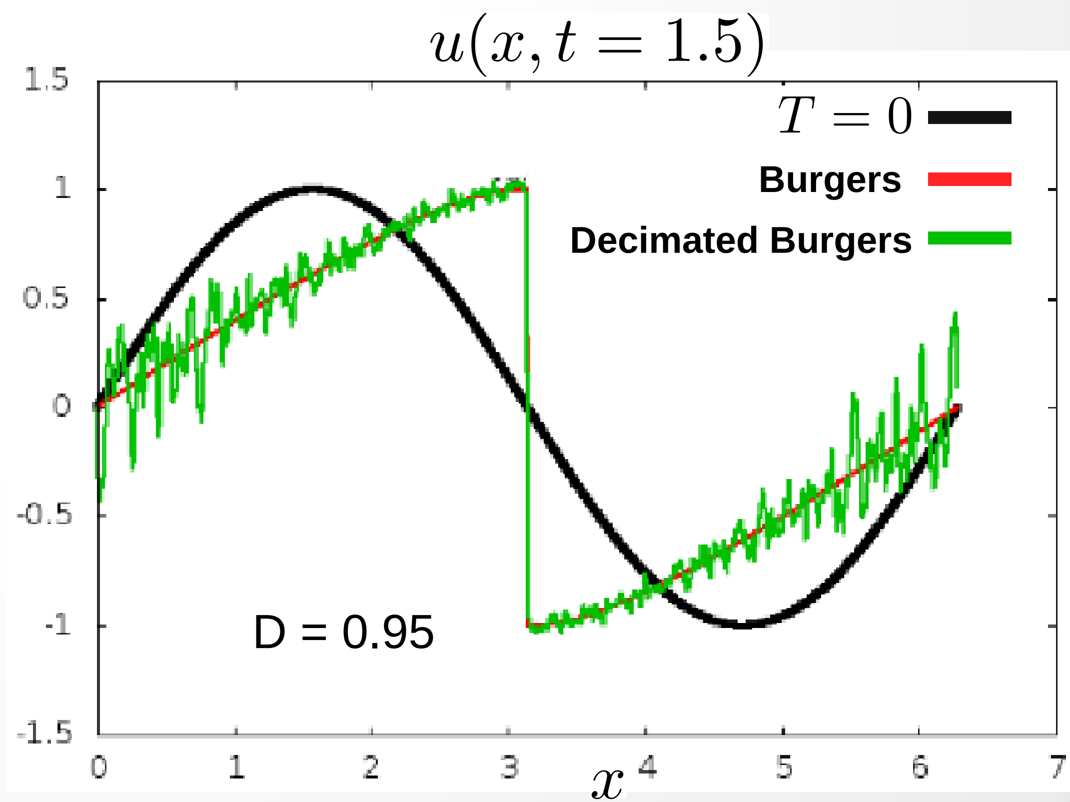
The decimation is Random but Quenched on time,

leaving on average $N(k) \sim k^D$ **active mode**

Galerkin truncation projection: $k < k_{max}$

⇓

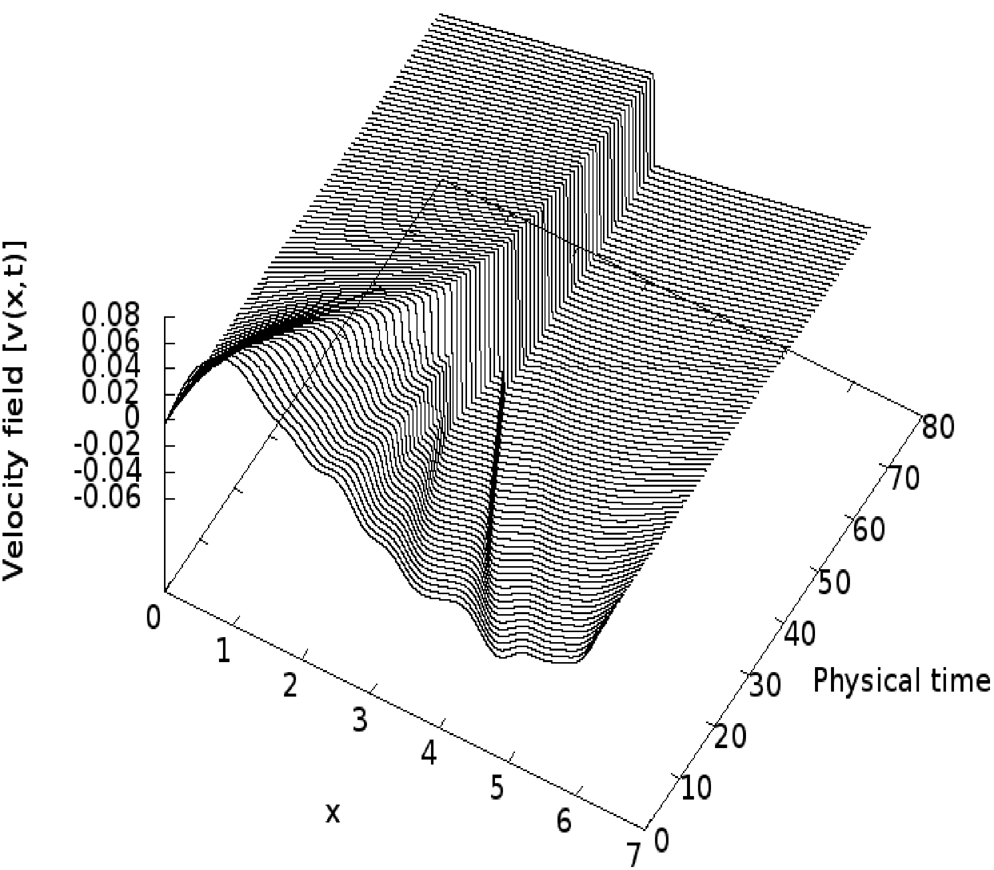
- **Finite number of d.o.f.**
- **Fractal dimension**

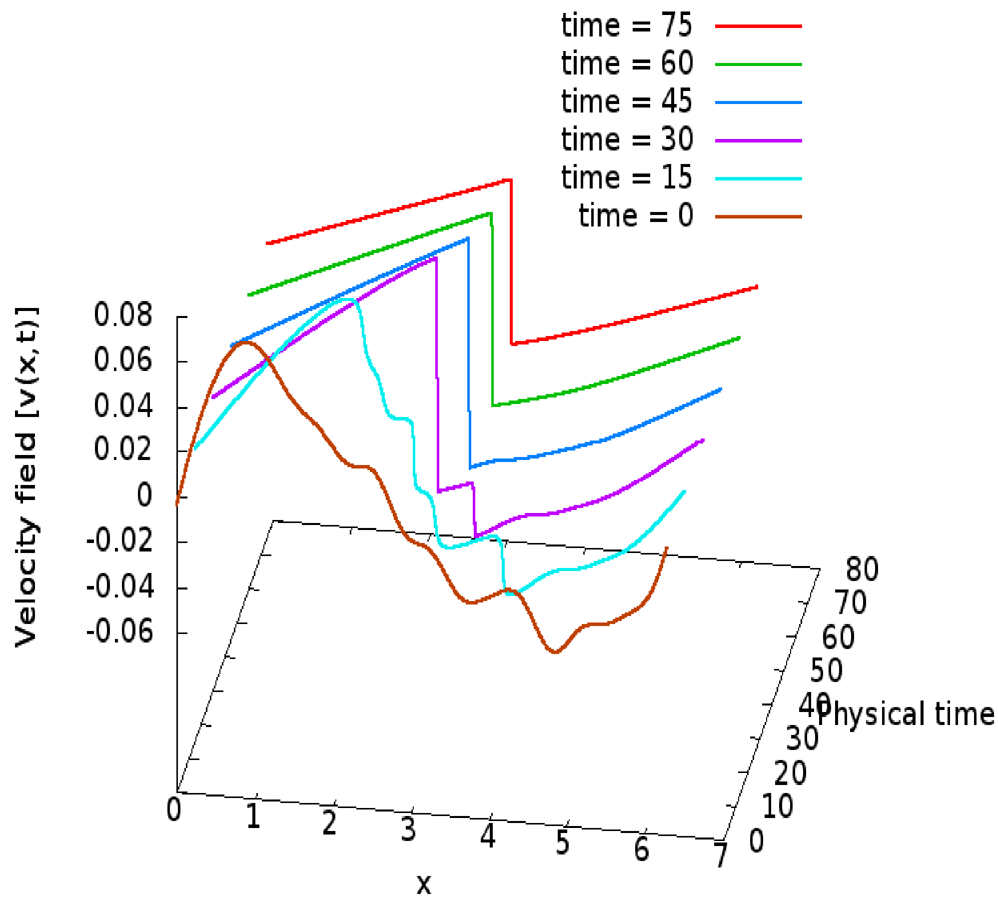


Real space velocity field evolution:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

$$Dim = 1$$

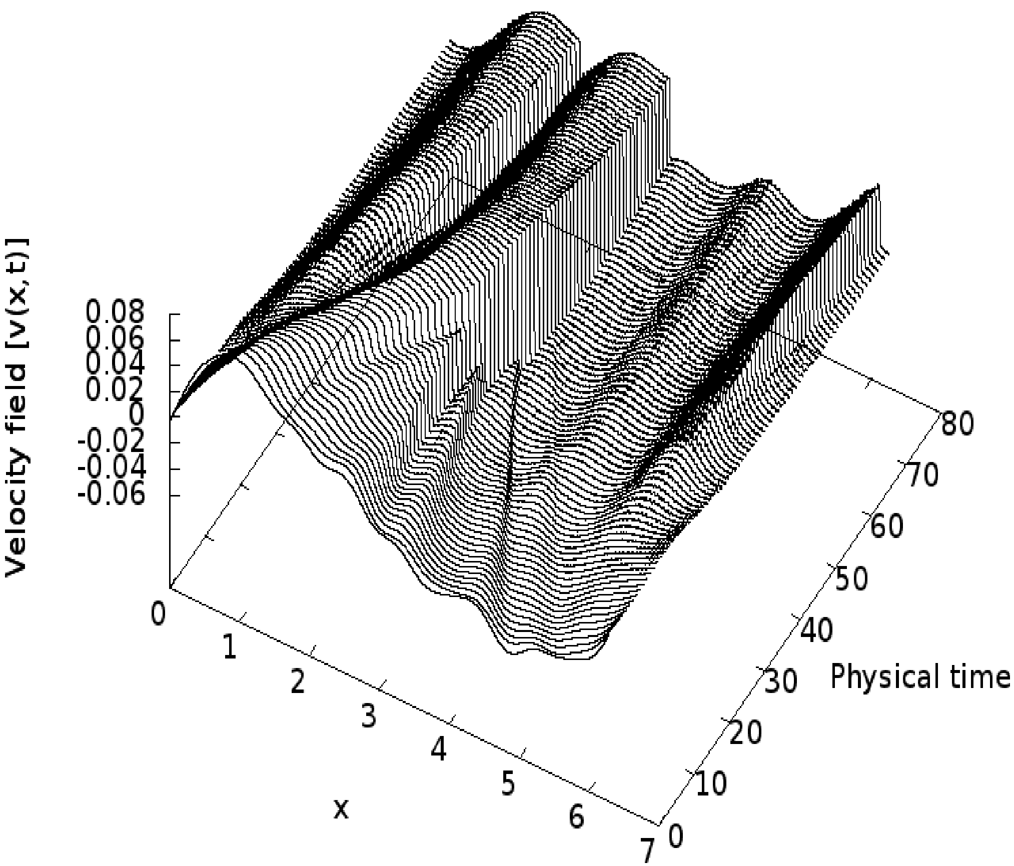


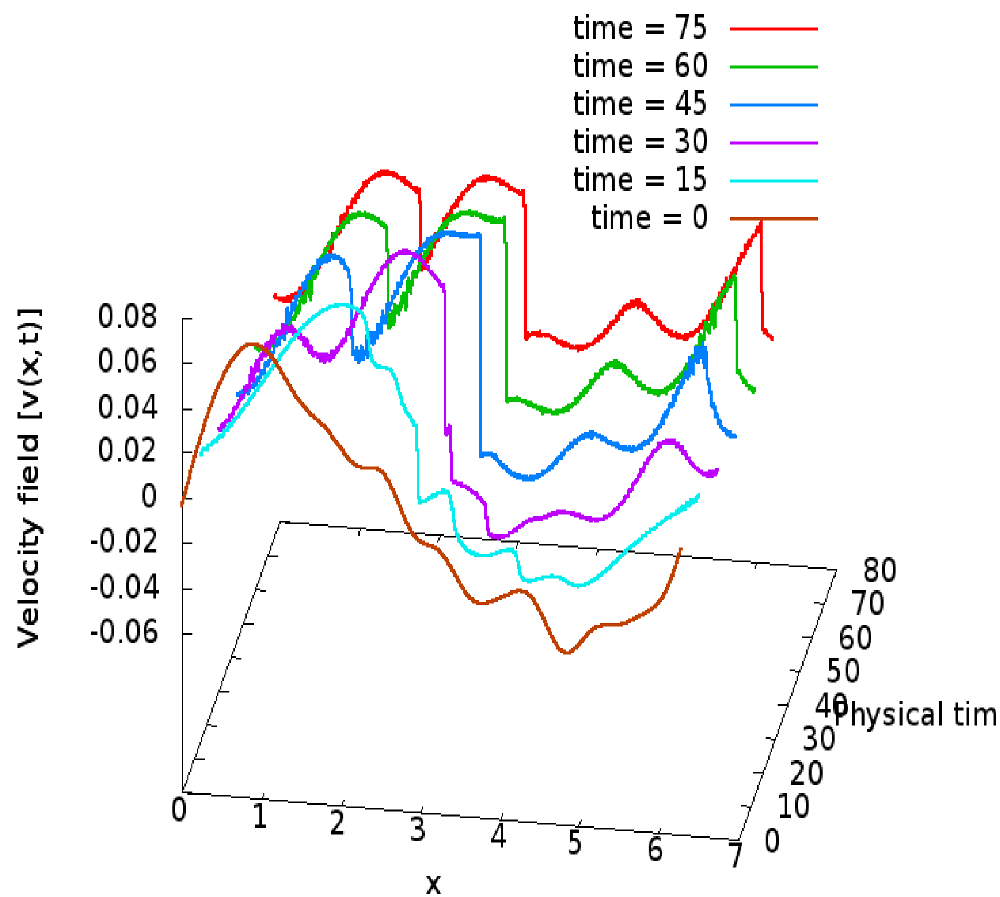


Real space velocity field evolution:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

$$Dim = 0.99$$

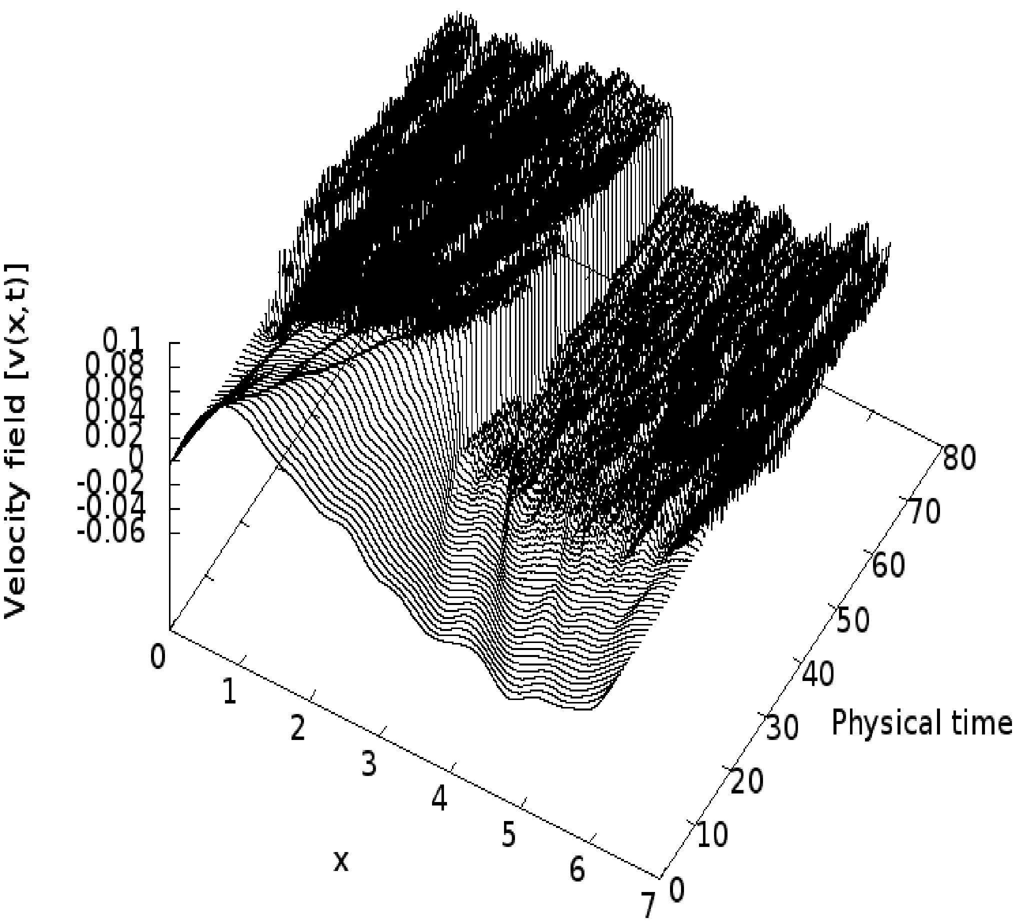


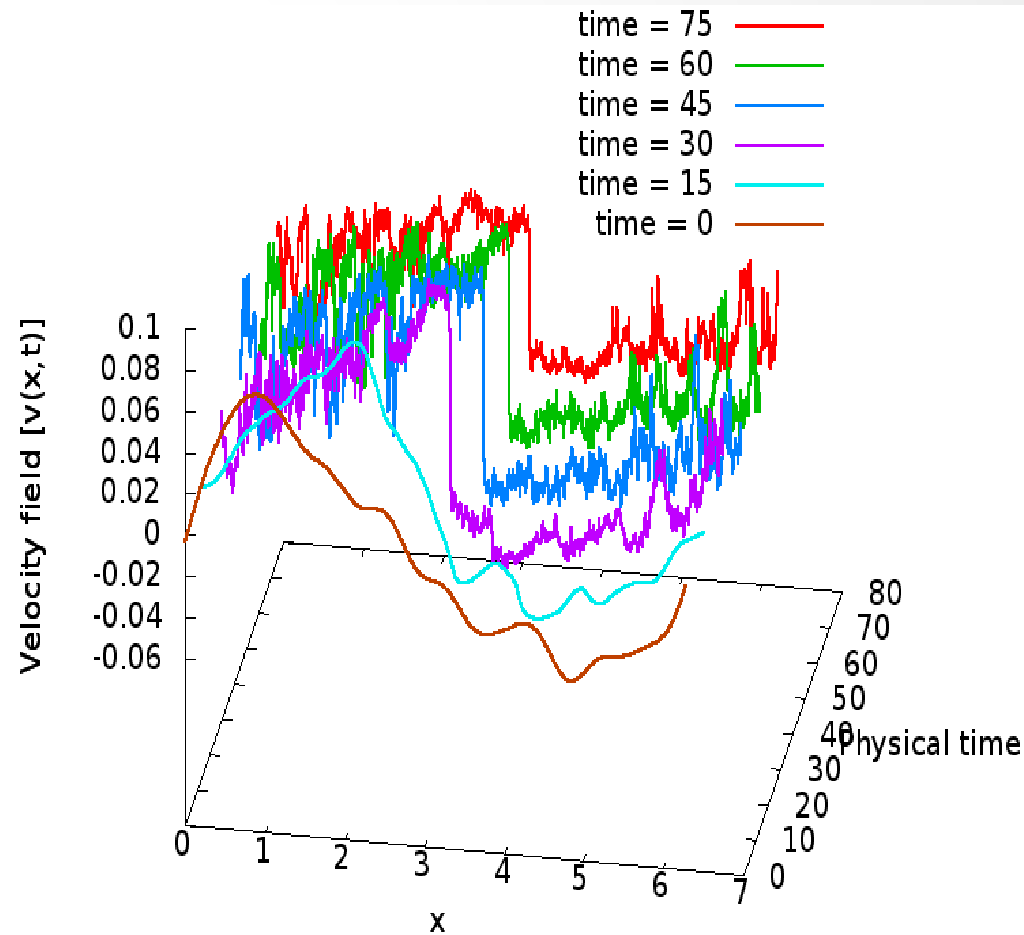


Real space velocity field evolution:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

$$Dim = 0.95$$

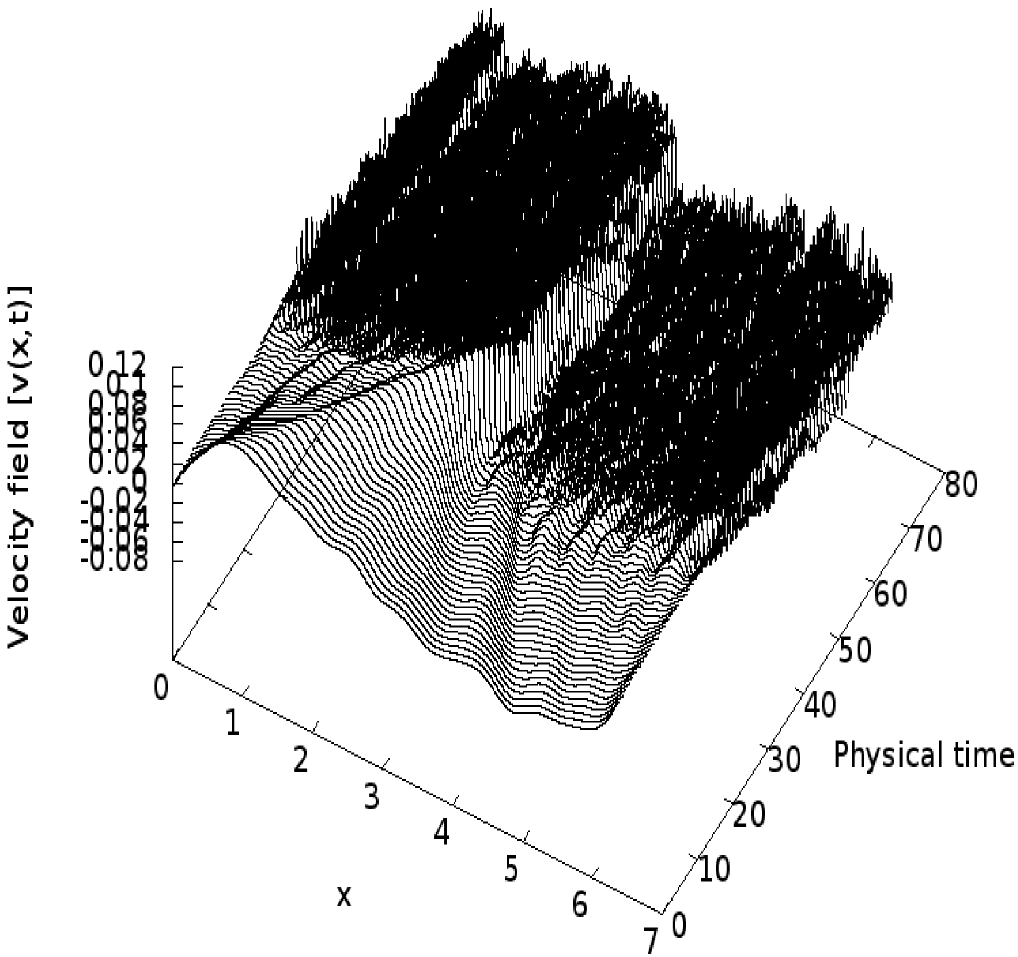


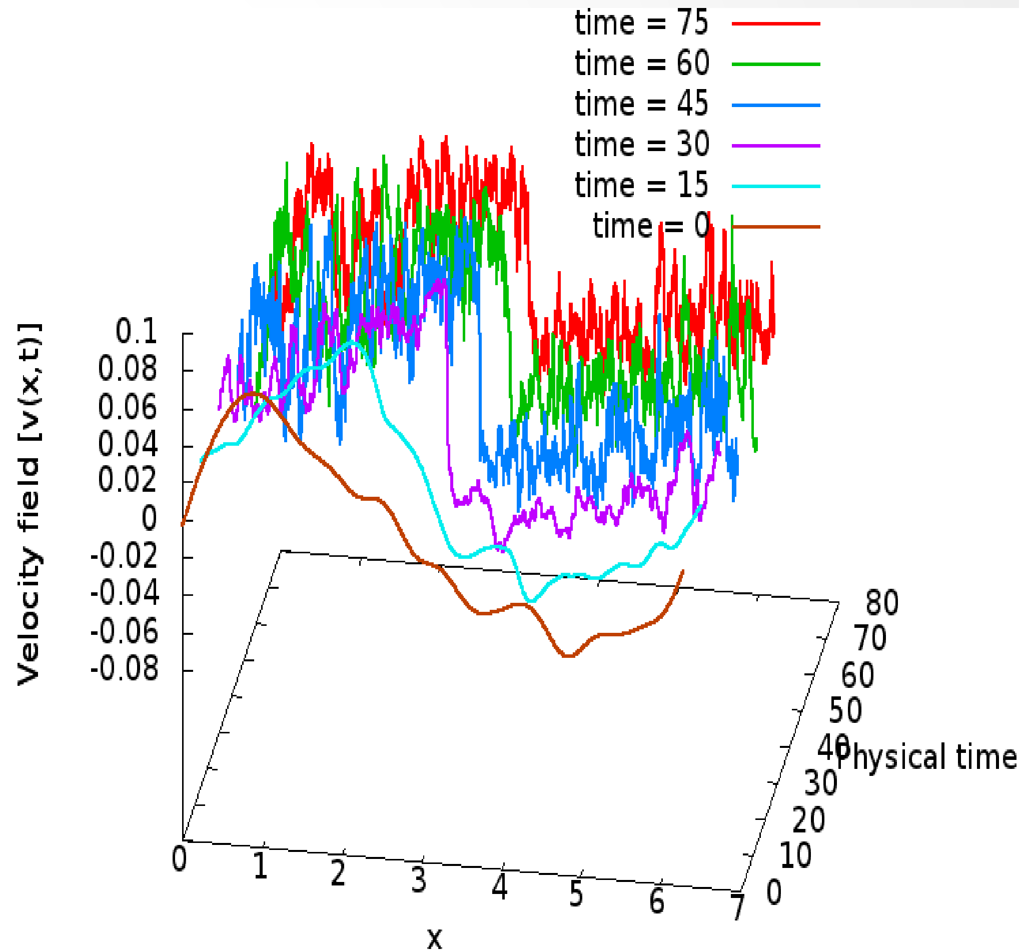


Real space velocity field evolution:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

$$Dim = 0.85$$





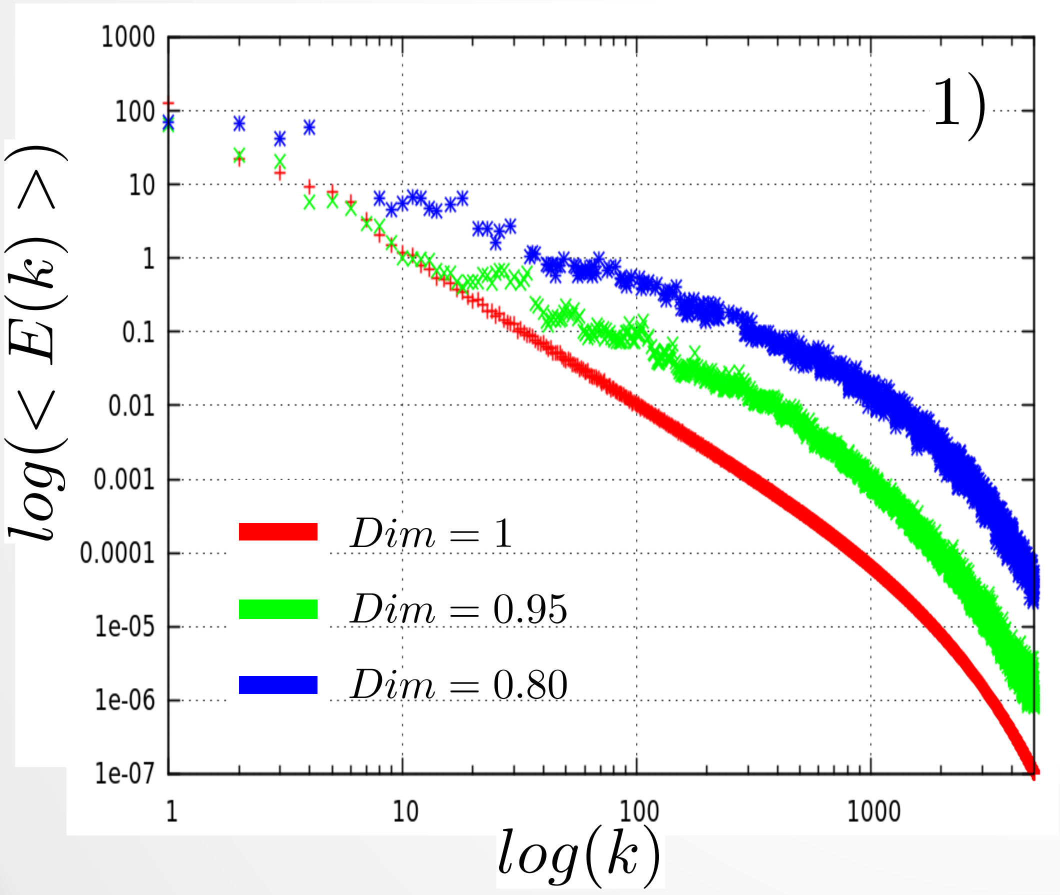
Decimated Energy Spectrum

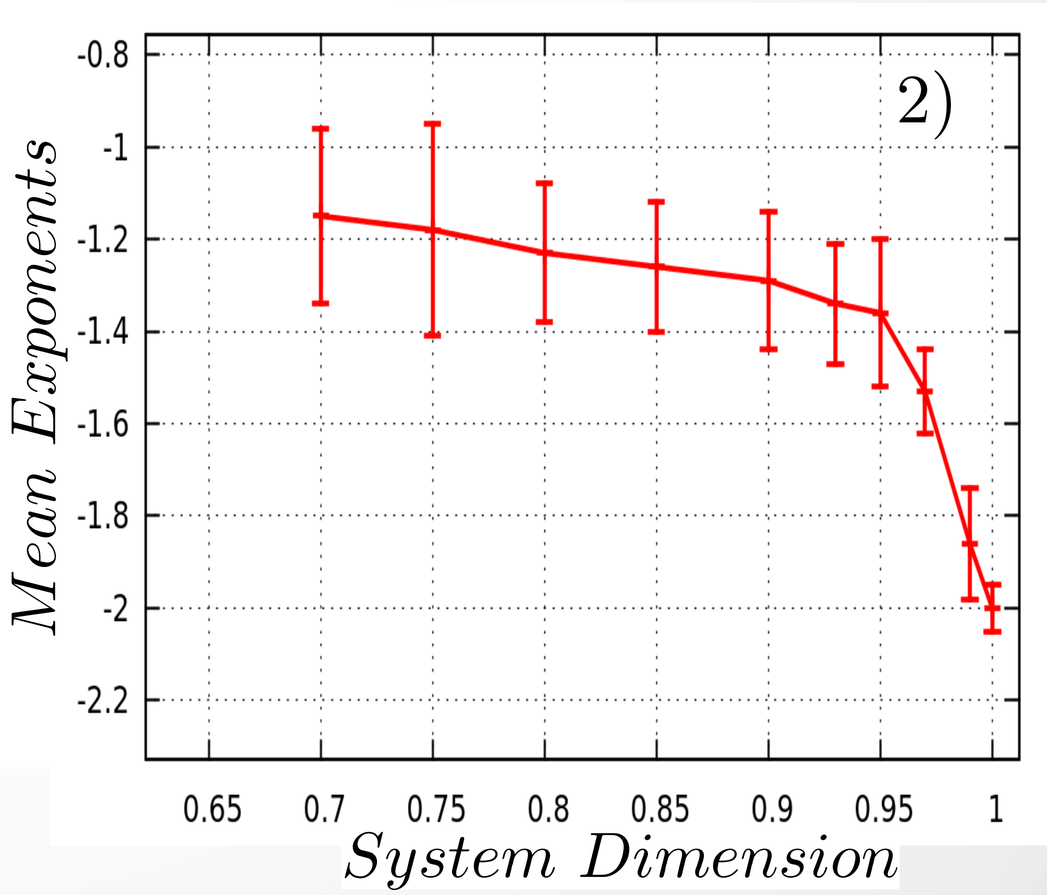
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

Mean spectra:

1) Single Mask: Averaged on time in the stationary state

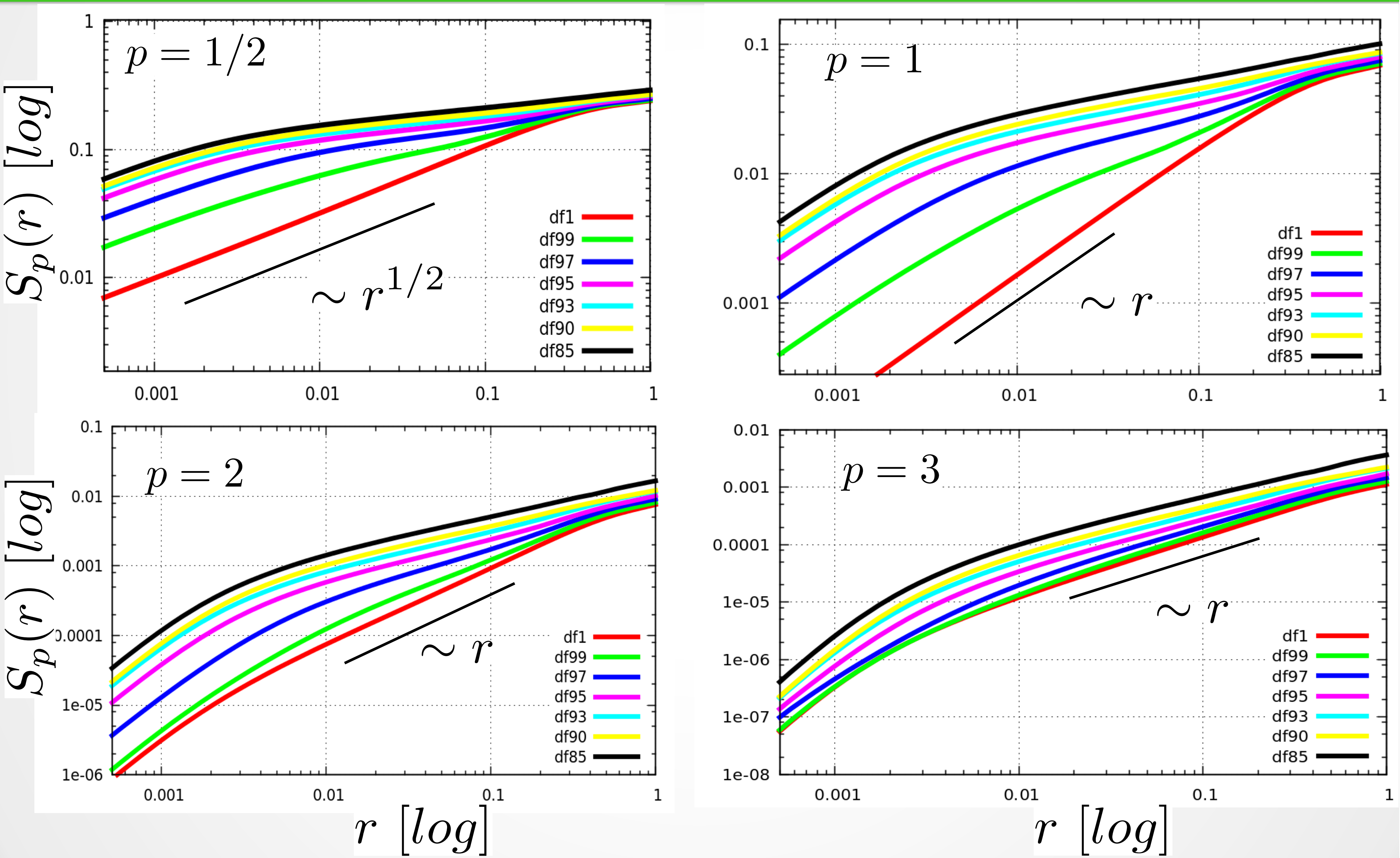
2) Mean Slope (32 masks) vs Fractal Dimension





Structure Functions

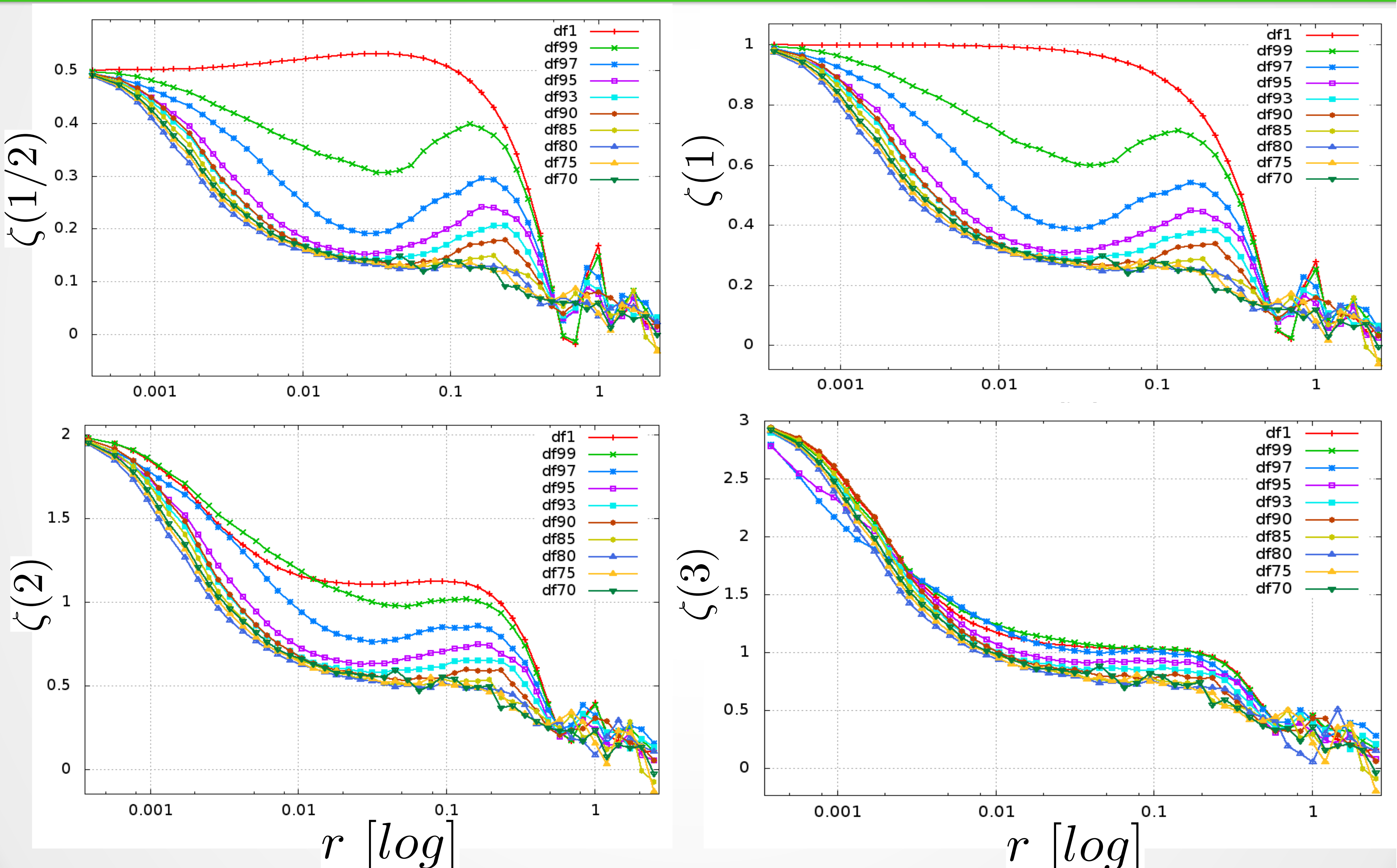
$$S_p(r) =< (\delta_r v)^p > \sim r^{\zeta(p)}$$



Structure Functions: Local Slope

$$S_p(r) = < (\delta_r v)^p > \sim r^{\zeta(p)}$$

$$\zeta(p) = \frac{\partial log(S_r(r))}{\partial log(r)}$$

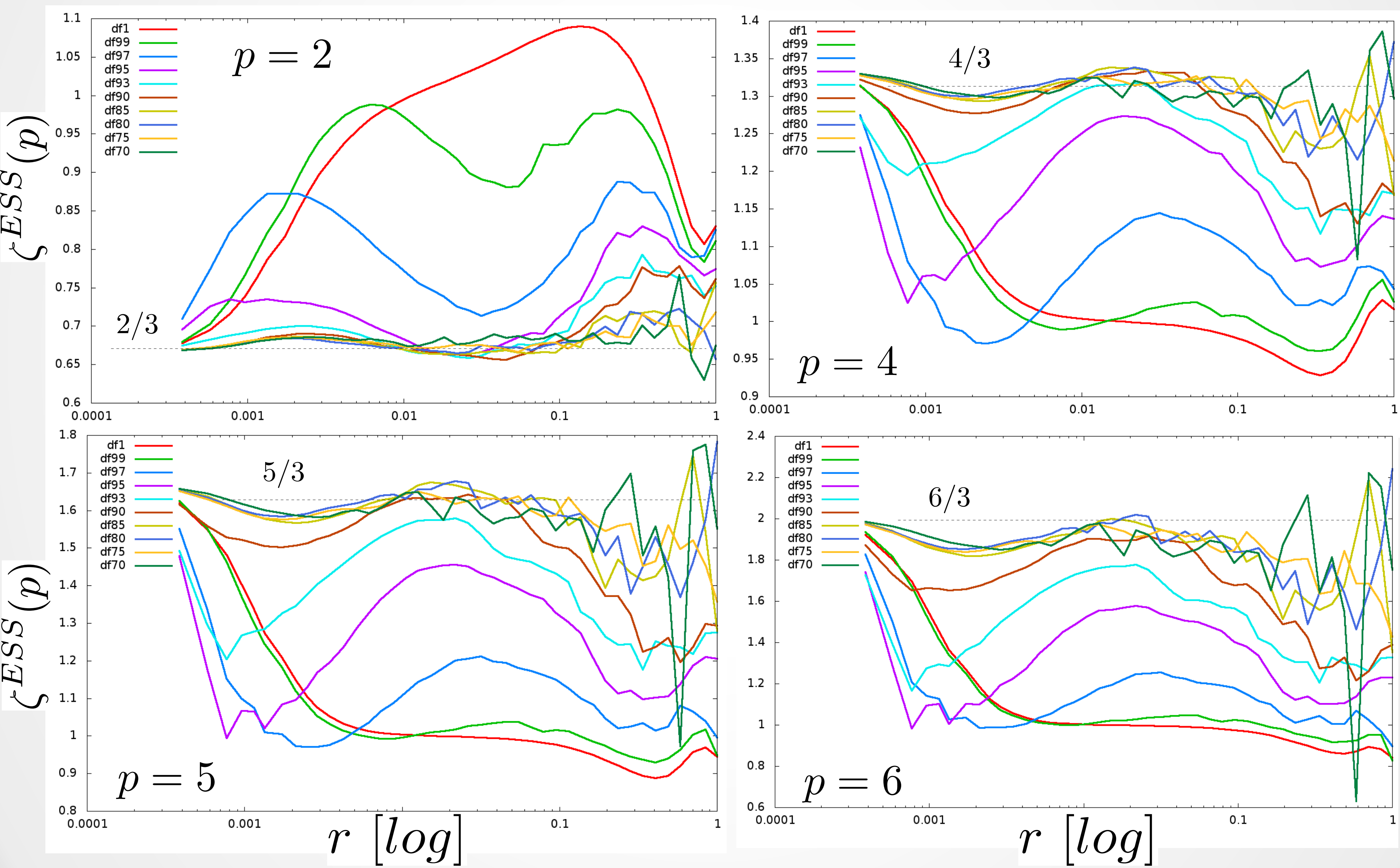


Extended Self Similarity (ESS):

$S_p(r) \ vs \ S_3(r)$

$$\zeta^{ESS}(p) = \frac{\partial log(S_p(r))}{\partial log(S_3(r))}$$

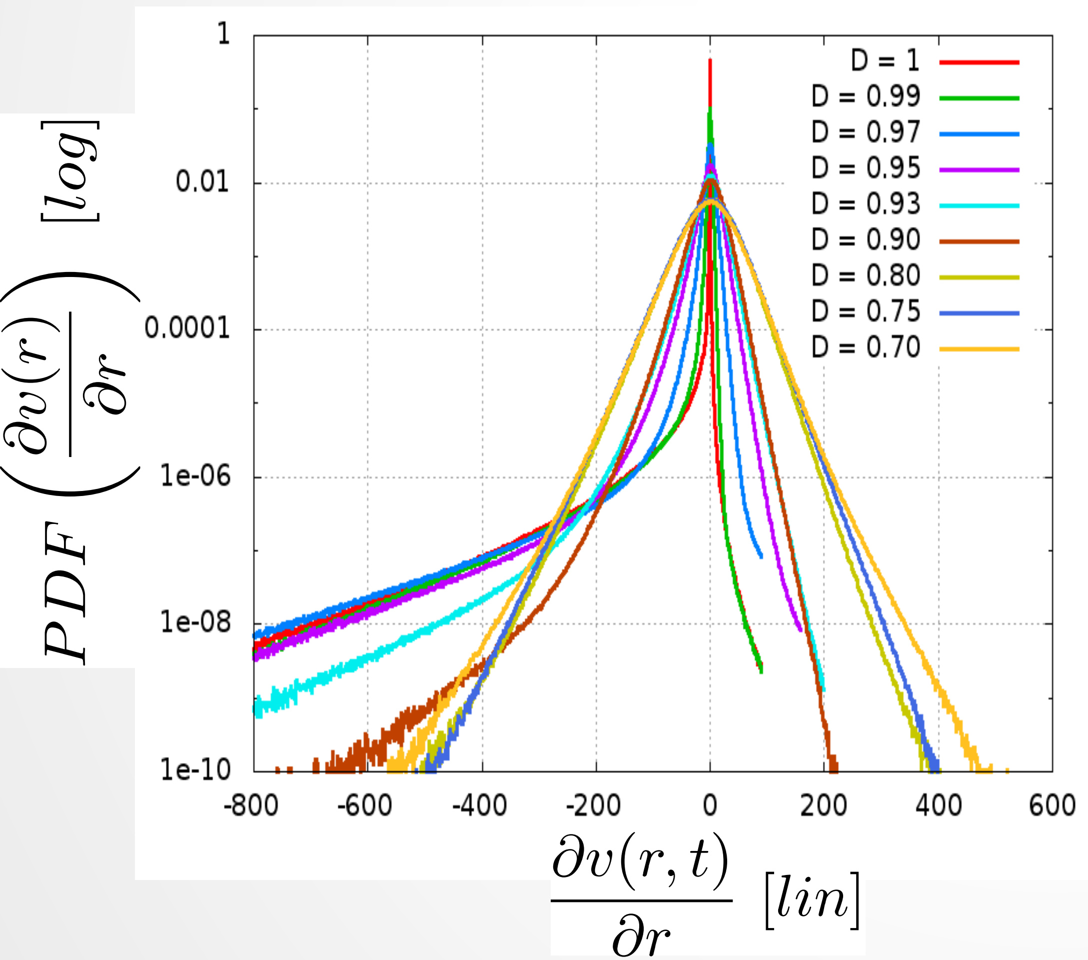
$$\zeta^{ESS}(p) = \frac{\zeta(p)}{\zeta(3)}$$

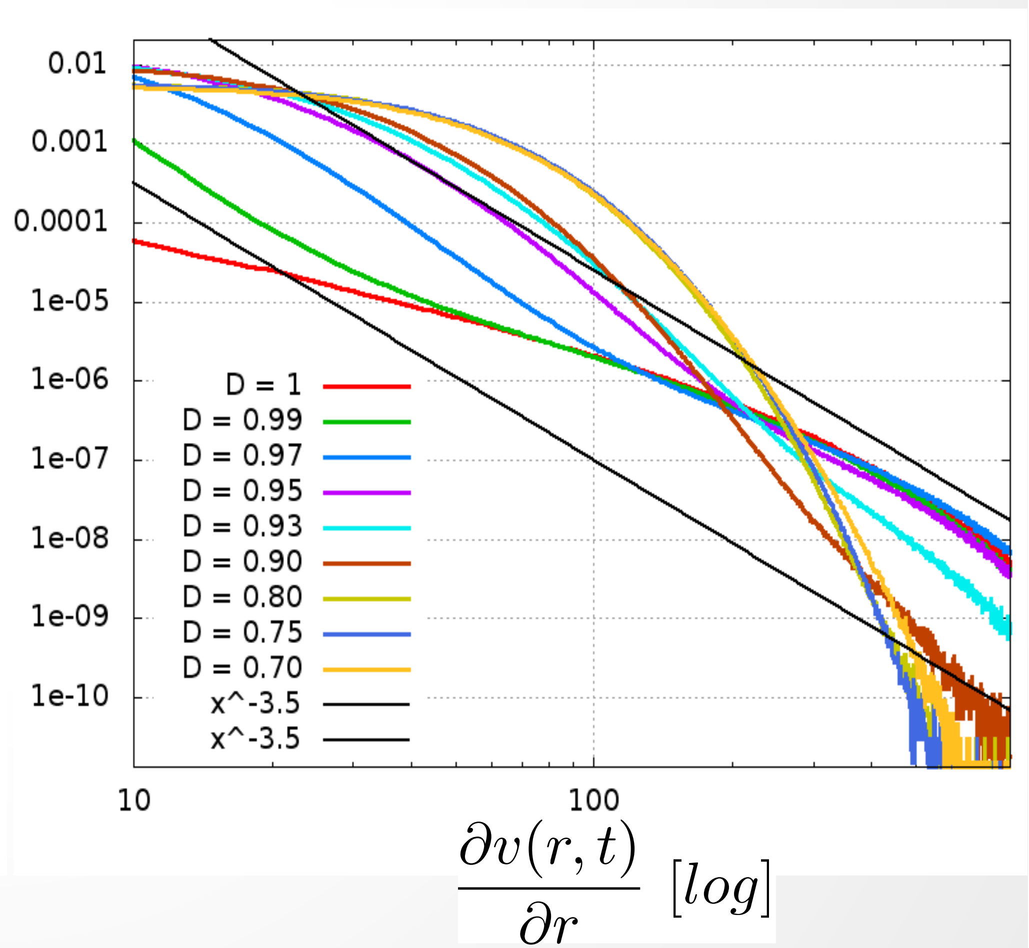


PDF of velocity gradients:

Khanin, Mazel
and Sinai, Probability Distribution Functions
for the Random Forced Burgers Equation,
(1997 Phys. Rev. Lett. 78, 1904)

$$PDF\left(\partial_r v(r)\right) \sim \partial_r v(r)^{-7/2}$$

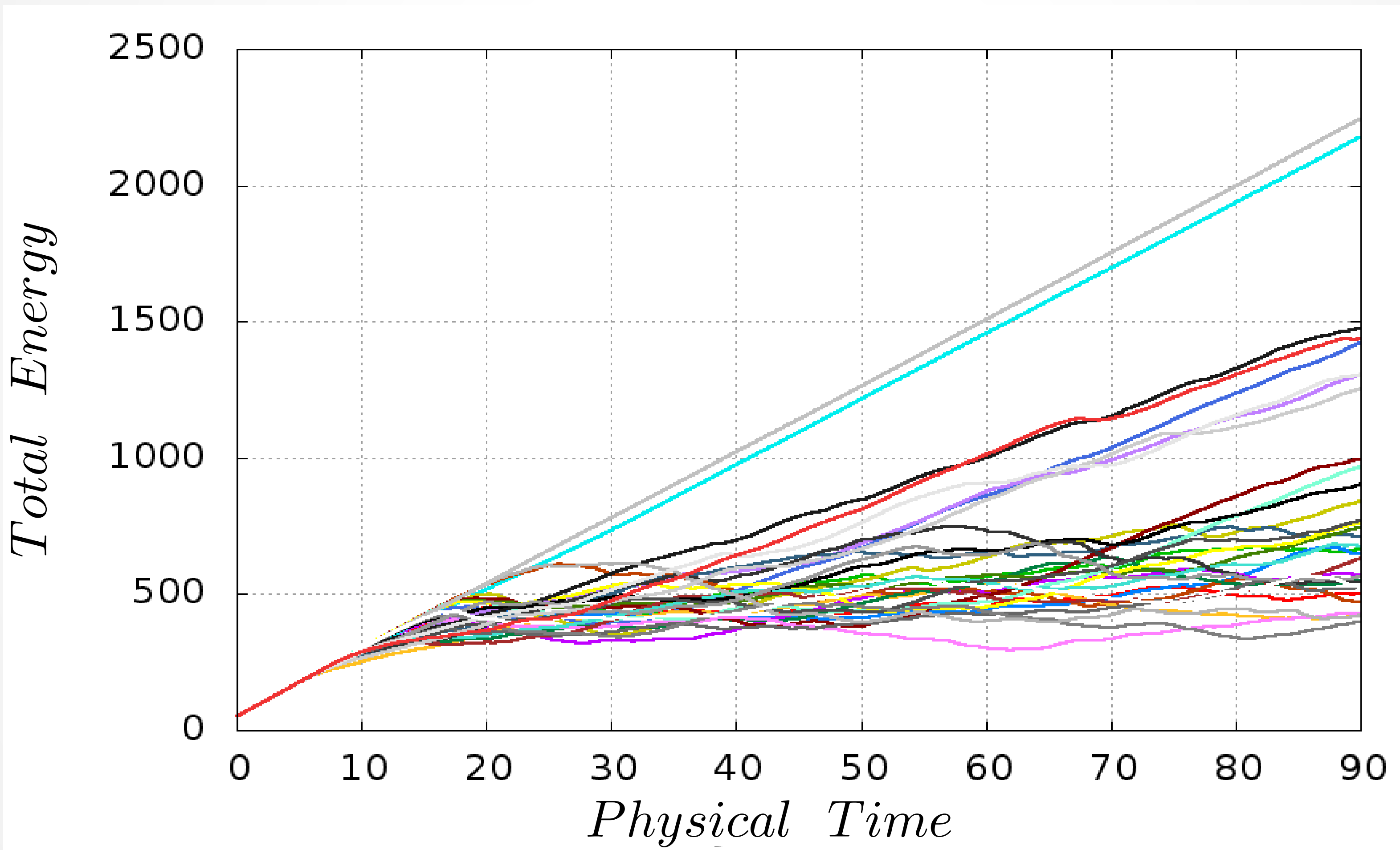




Non Self-Averaging Problem

Total energy evolution: **different masks**

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} + f_{deterministic}(x,t)$$

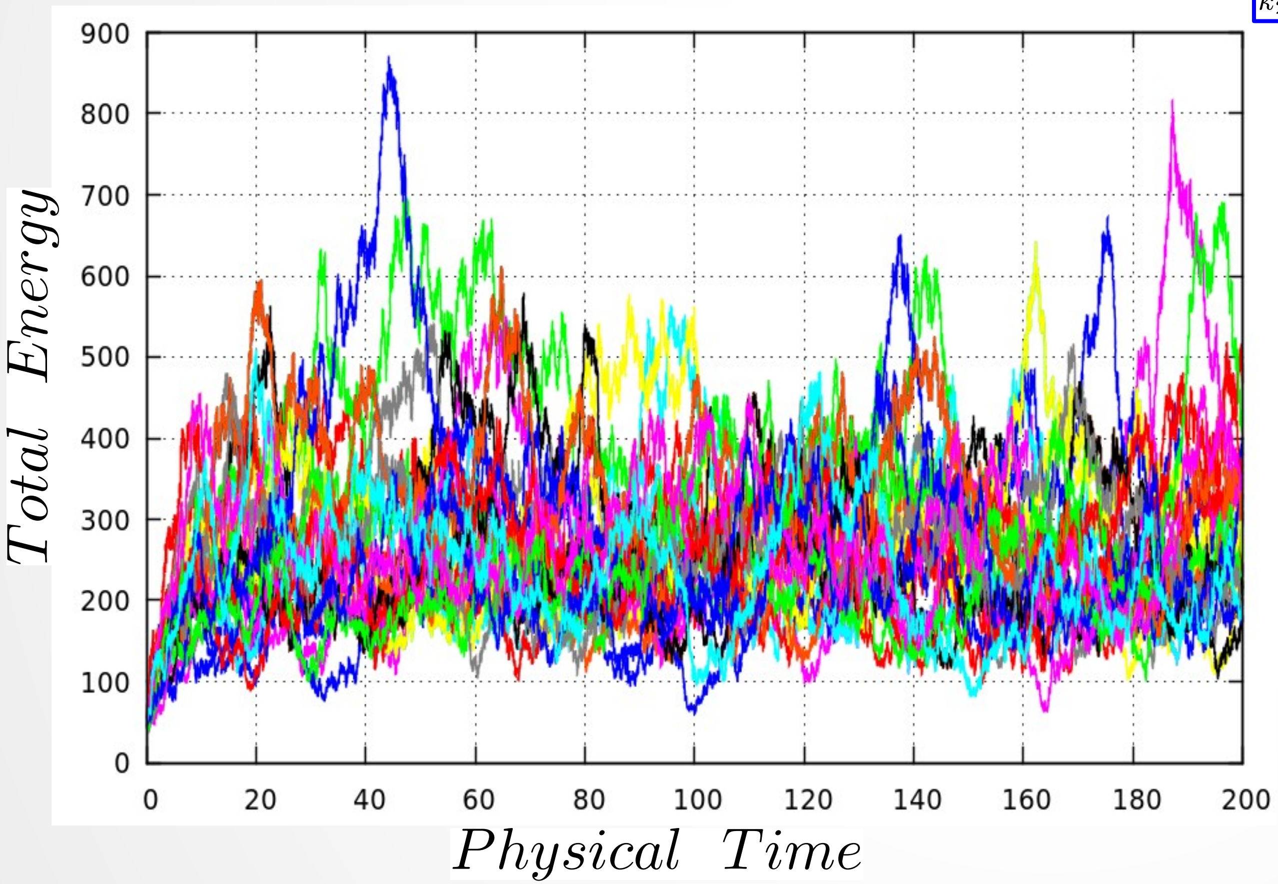


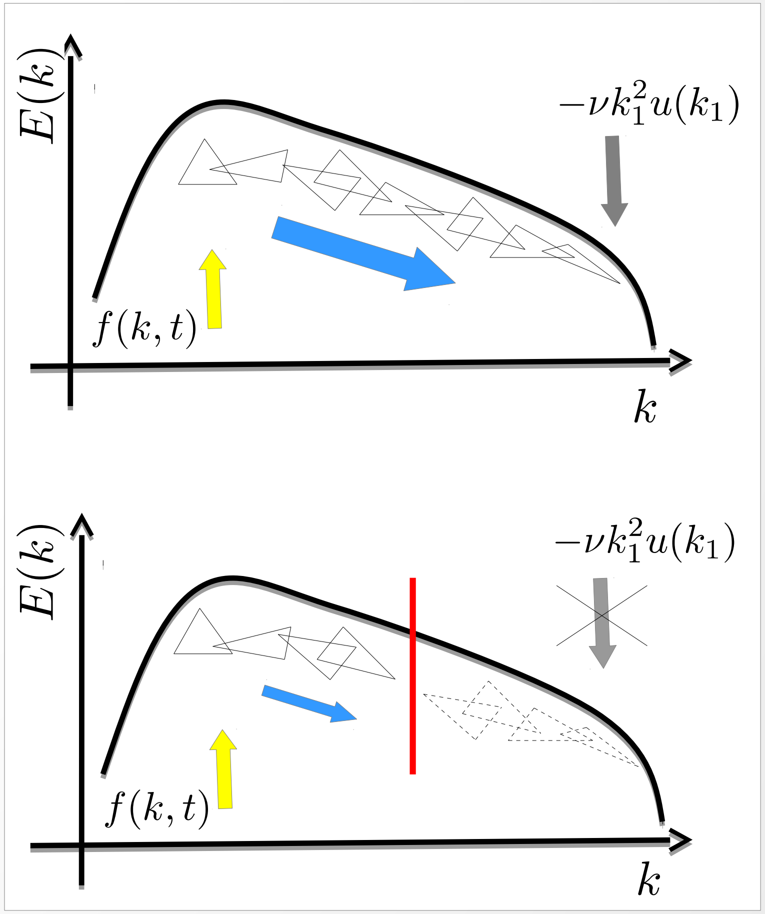
Block in the **energy transfer**

..solution

Total energy evolution: **different masks**

$$\frac{\partial u(k_1,t)}{\partial t} = \boxed{\sum_{k_2+k_3=k_1} \Pi(u(k_2),u(k_3))} - \nu k_1^2 u(k_1) + f_{stoc}(k,t)$$





Block in the **energy transfer**

Conclusions:

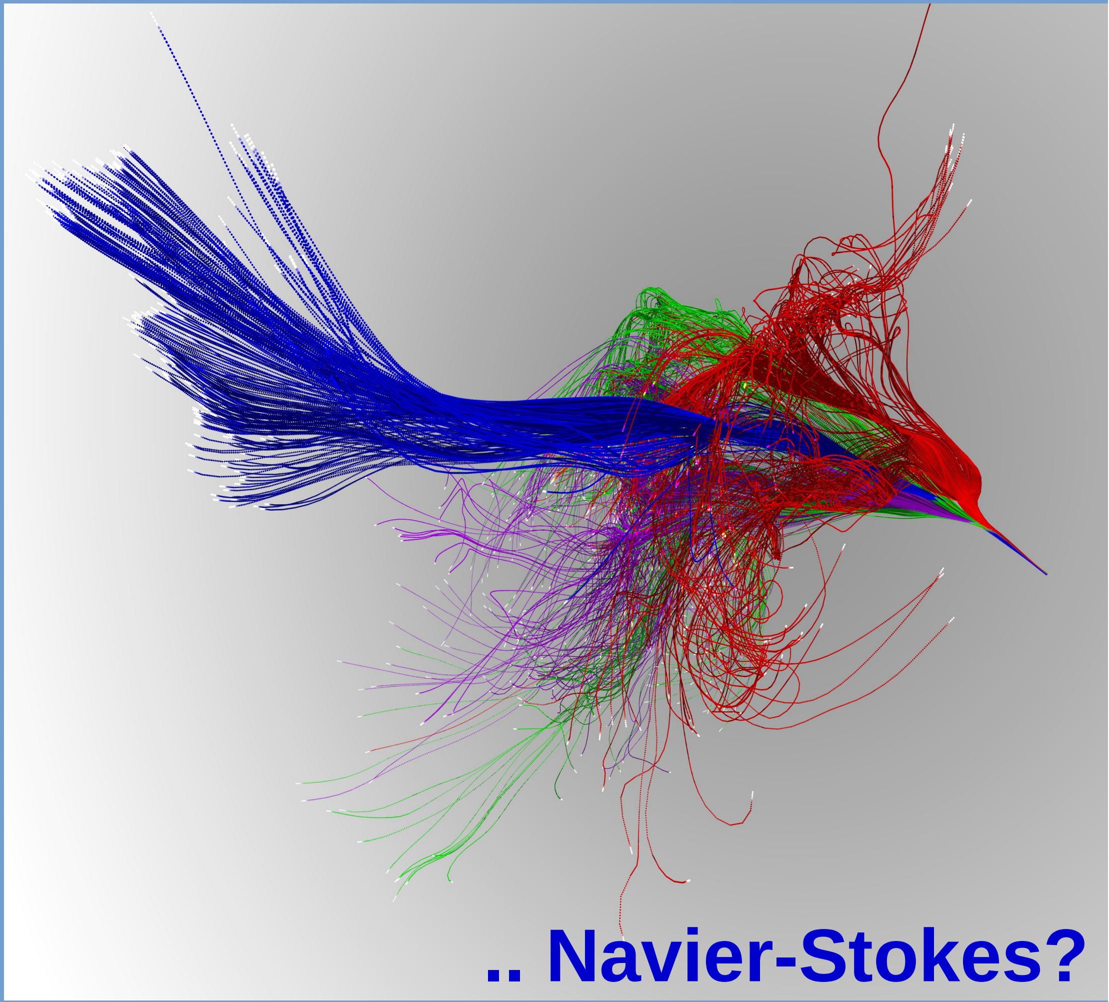
1) Small scales structures appear in the real space velocity field when the system is evolved in a dimension less than 1

2) Differences in the statistic have been found among the different dimensions

- Structure functions
- Pdfs of Gradients

3) The system seems to be no longer intermittent with the decreasing of the dimension starting from a value close to 0.90

4) We need to understand the relation between the scaling laws of 2° order structure functions and energy spectra



Intermittecy on Burgers' equation

..BIFRACTAL MODEL

$$\frac{\delta v_\ell(r)}{v_0} \sim \begin{cases} \left(\frac{\ell}{\ell_0}\right)^{h_1}, & r \in \mathscr{S}_1,\ \dim \mathscr{S}_1 = D_1 \\ \left(\frac{\ell}{\ell_0}\right)^{h_2}, & r \in \mathscr{S}_2,\ \dim \mathscr{S}_2 = D_2 \end{cases}$$

$$\frac{\langle \delta v_\ell^p \rangle}{v_0^p} \propto \left(\frac{\ell}{\ell_0}\right)^{ph_1} \left(\frac{\ell}{\ell_0}\right)^{1-D_1} + \left(\frac{\ell}{\ell_0}\right)^{ph_2} \left(\frac{\ell}{\ell_0}\right)^{1-D_2}$$

Probability *to be within a distance l of the set* $\mathscr{S}$

$D_1=0$; $h_1=0$ ← isolated shock

$D_2=1$; $h_2=1$ ← smooth ramps

$$\frac{\langle \delta v_\ell^p \rangle}{v_0^p} \propto \left(\frac{\ell}{\ell_0}\right)^{1} + \left(\frac{\ell}{\ell_0}\right)^{p}$$